非線形コンクリート特論（Nonlinear Behavior of Concrete and Concrete Members）

(2) 変形とひずみの関係 Deformation-Strain Relationship

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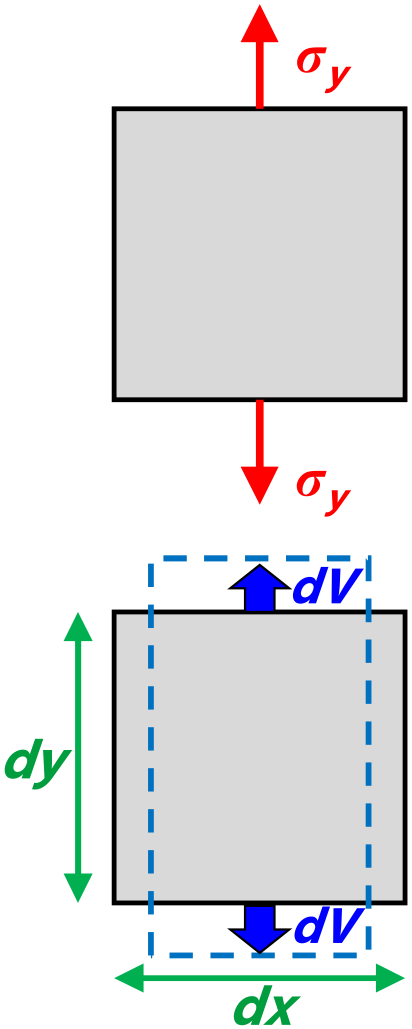
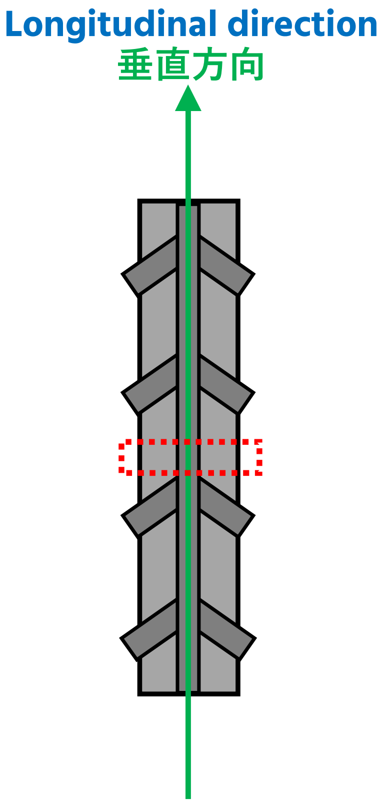
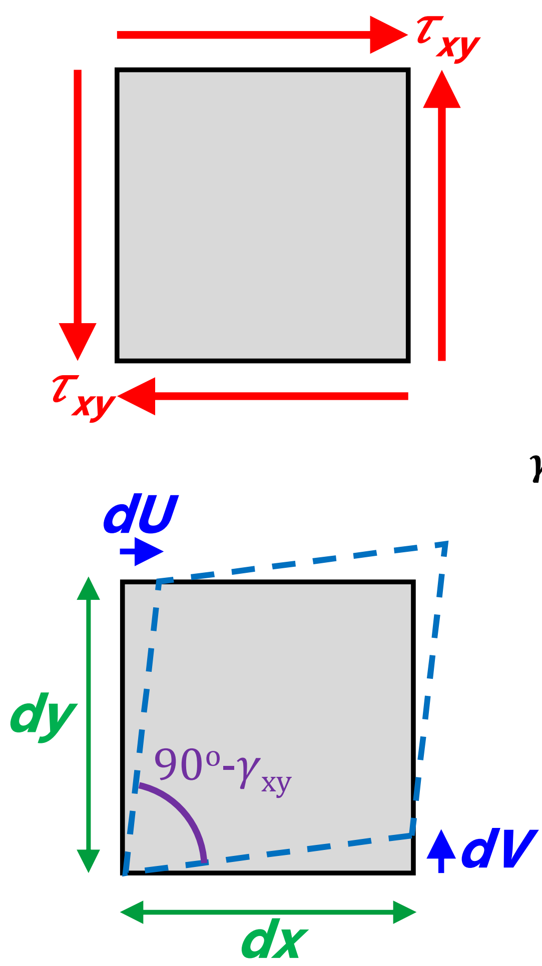
前回の授業・Previous Lecture

Normal strain
垂直ひずみ

$$\varepsilon_{yy} = \frac{dV}{dy}$$



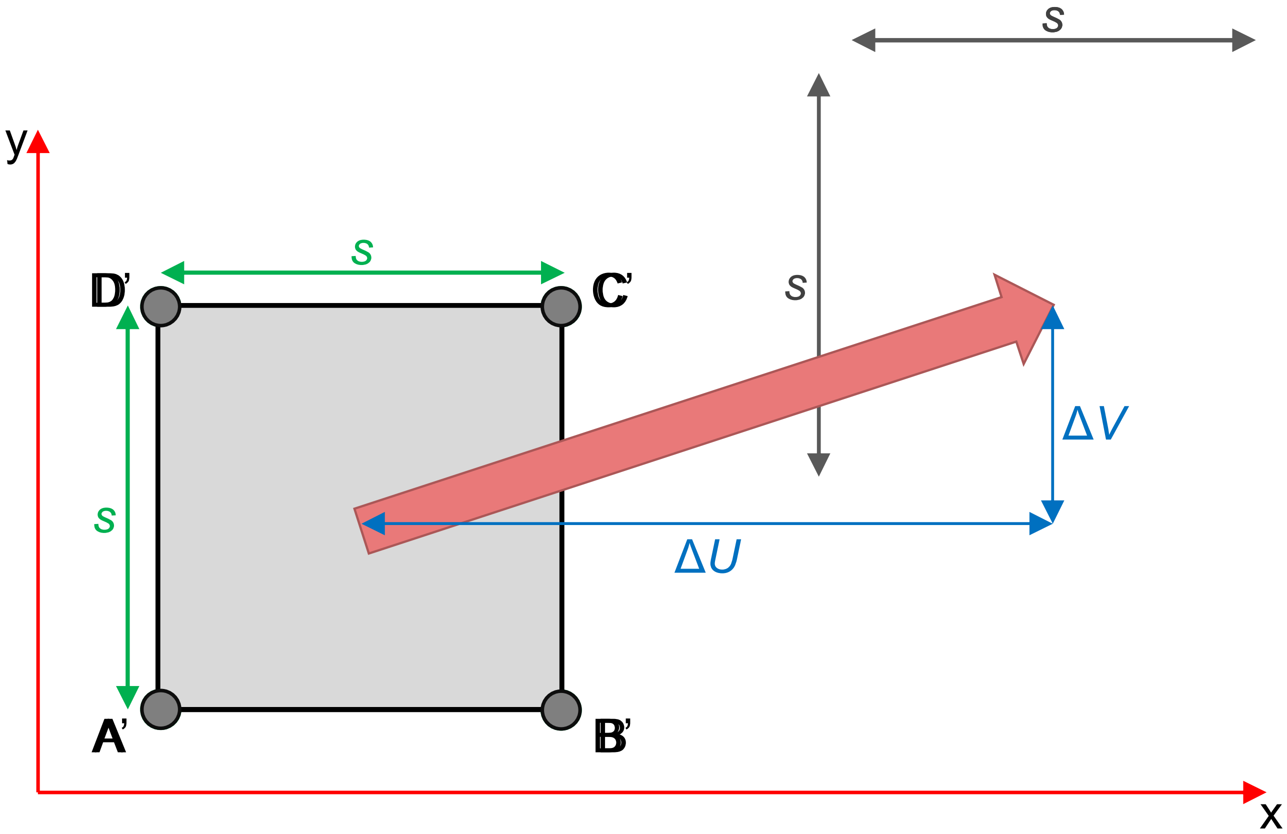




Shear strain
せん断ひずみ

$$\gamma_{xy} = \frac{dU}{dy} + \frac{dV}{dx}$$

変位・Displacement



平行移動　Translation

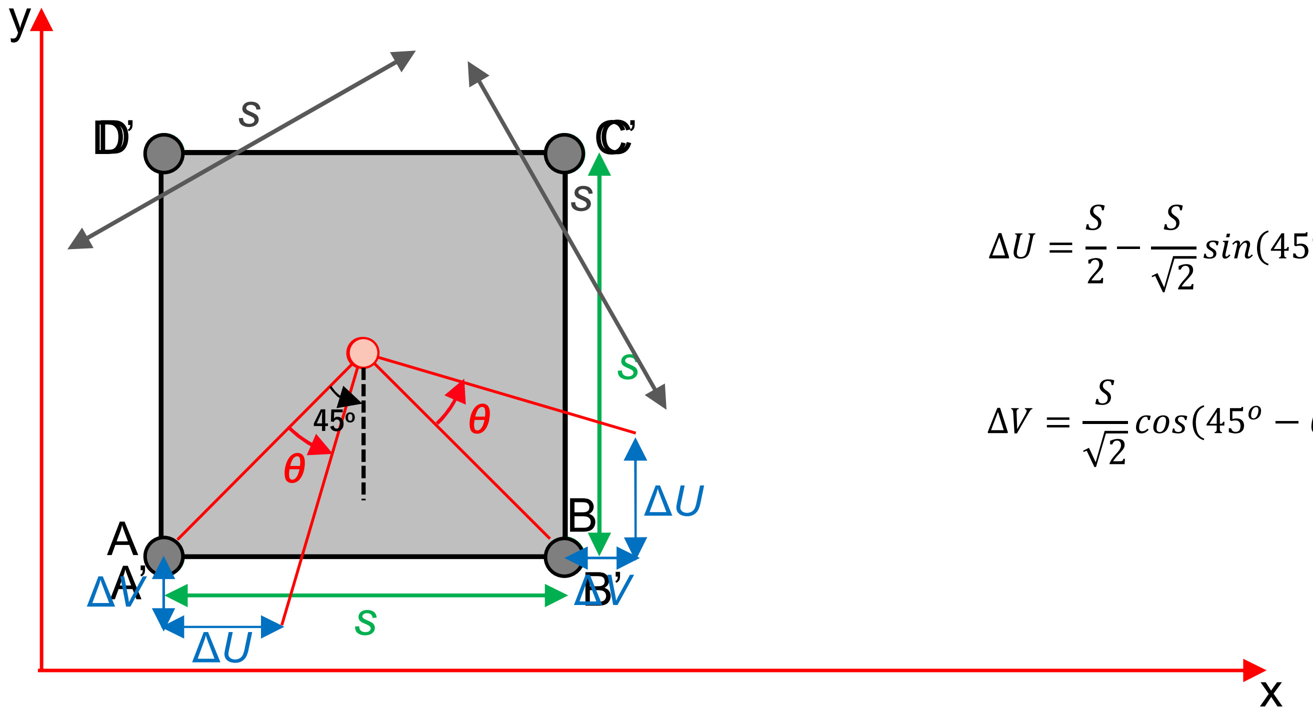
$U(A')=U(A)+\Delta U$ and $U(B')=U(A)+S+\Delta U$

$V(A')=V(A)+\Delta V$ and $V(B')=V(A)+S+\Delta V$

$$L'_{AB}=\sqrt{\left(U(B')-U(A')\right)^2+\left(V(B')-V(A')\right)^2}=S$$

No change along other edges either – no strain

変位・Displacement



$$\Delta U = \frac{S}{2} - \frac{S}{\sqrt{2}} sin(45^o - \theta)$$

$$\Delta V = \frac{S}{\sqrt{2}} cos(45^o - \theta) - \frac{S}{2}$$

回転運動 Rotation

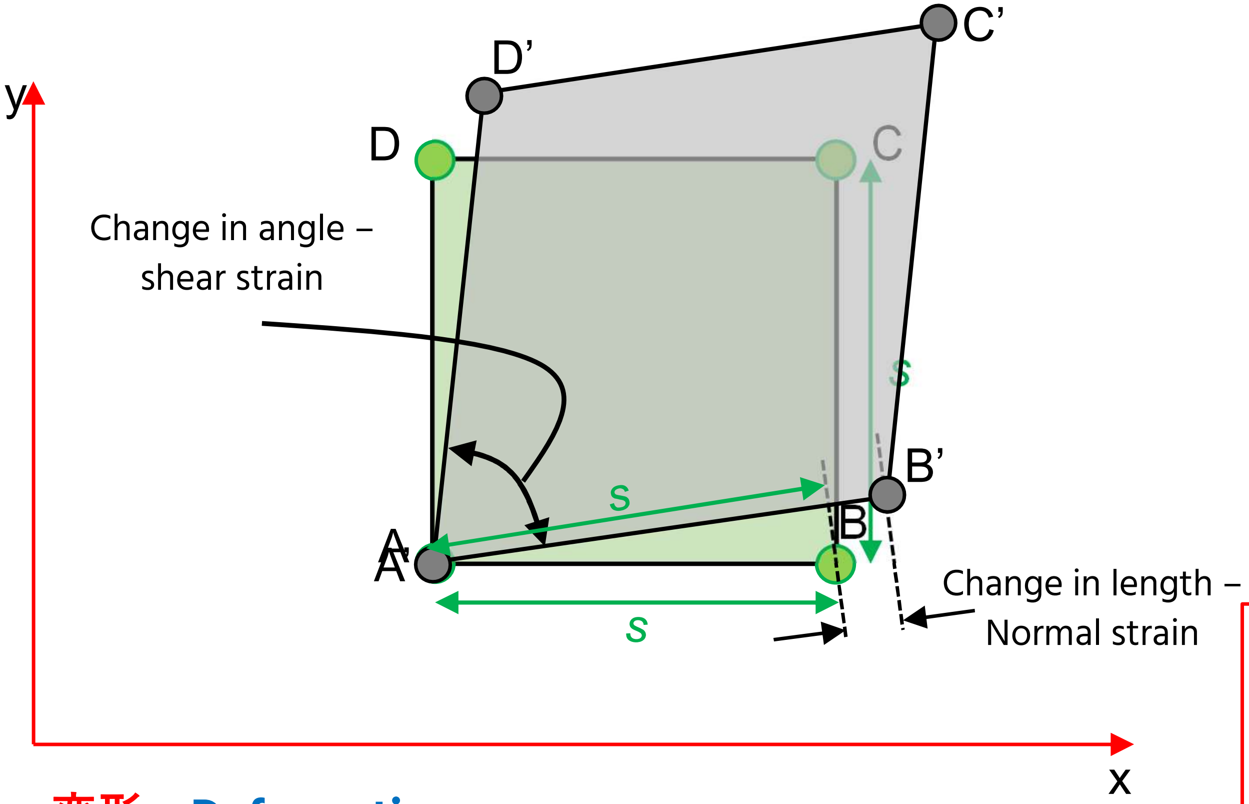
U(*A*')=*U*(*A*)+Δ*U* and *U*(*B*')= *U*(*A*)+*S*+Δ*V*

V(*A*')=*V*(*A*)-Δ*V* and *V*(*B*')=*V*(*A*)+*S*+Δ*U*

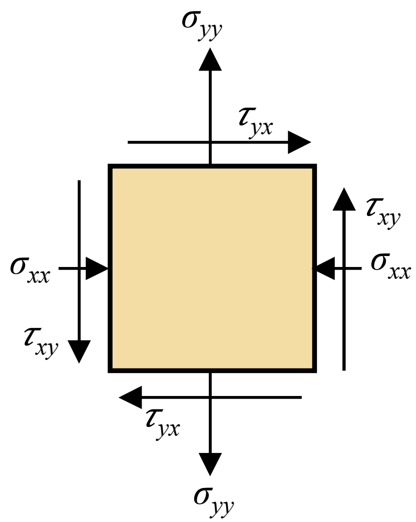
$$L'_{AB} = \sqrt{\left(U(B') - U(A')\right)^2 + \left(V(B') - V(A')\right)^2} = S$$

No change along other edges either – no strain

変位・Displacement



変形　**Deformation**

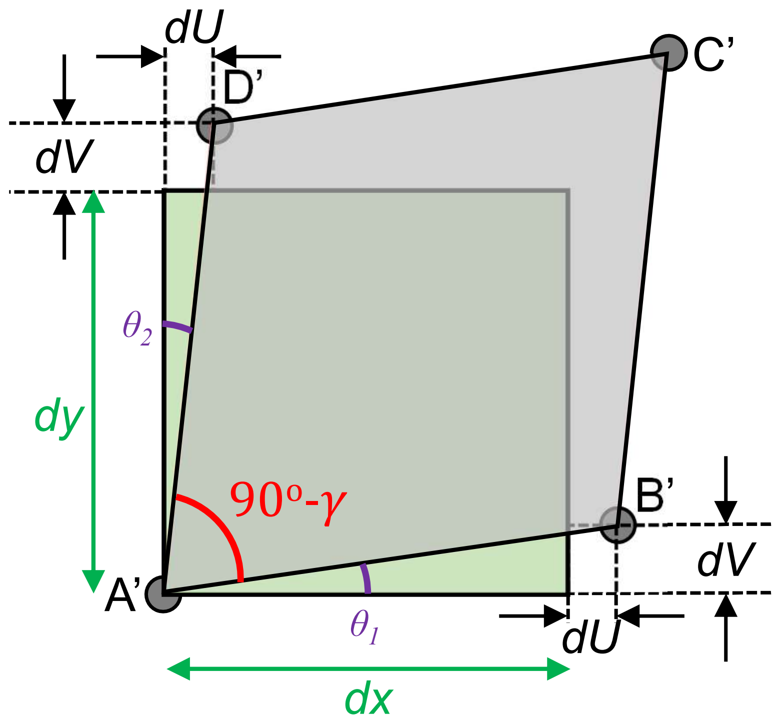


τ_{AB}

A軸に直角面の
B方向せん断応力

Shear stress acting in B direction on face perpendicular to A axis

変形・Deformation



垂直ひずみ　Normal strain

$$\varepsilon_{xx} = \frac{(dx+d\ \ )/cos\theta_1 - dx}{dx} \approx \frac{dU}{dx} \text{ if } \theta_1 \text{ is small}$$

Likewise $\varepsilon_{yy} \approx \frac{dV}{dy}$

せん断ひずみ　Shear strain

$$\gamma = tan^{-1}\left(\frac{dU}{dy+dV}\right) + tan^{-1}\left(\frac{dV}{dx+dU}\right) \approx \frac{dU}{dy} + \frac{dV}{dx}$$

If angle is small and $dy >> dV$ and $dx >> dU$

せん断ひずみ表記・Shear Strain Notation

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- これまで、せん断ひずみγ_{xy}を直角からの変化量と定義した。これは**工学せん断ひずみ**と呼ぶ。
  Up until now, we defined shear strain γ_{xy} as the change in angle. This is known as **ENGINEERING** shear strain.

- しかし、用途によっては、せん断ひずみを**テンソルひずみ**ε_{xy}で示すこともある。
  However, in some applications shear strain may be represented using **TENSOR** strain, ε_{xy}

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{dU}{dy} + \frac{dV}{dx}\right) = \frac{\gamma_{xy}}{2} \quad \text{or in generic form} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{dU_i}{dx_j} + \frac{dU_j}{dx_i}\right)$$

- この形式は、垂直ひずみ（*i*=*j*の場合）にも適用できるように使用されている。
  The form is used so that it can also be applied to normal strain (i.e., *i*=*j*)

$$\varepsilon_{ii} = \frac{1}{2}\left(\frac{dU_i}{dx_i} + \frac{dU_i}{dx_i}\right) = \frac{dU_i}{dx_i}$$

ひずみテンソル
Strain tensor

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}$$

- $\boldsymbol{\gamma_{ij} = \frac{dU_i}{dx_j} + \frac{dU_j}{dx_i}}$ **工学せん断ひずみ・engineering shear strain**

  $\boldsymbol{\varepsilon_{ij} = \frac{1}{2}\left(\frac{dU_i}{dx_j} + \frac{dU_j}{dx_i}\right)}$ **テンソルせん断ひずみ・tensor shear strain**

3D変形ーひずみ・3D Deformation-Strain

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- Normal strains

$$\varepsilon_{xx} = \frac{dU}{dx}$$

$$\varepsilon_{yy} = \frac{dV}{dy}$$

$$\varepsilon_{zz} = \frac{dW}{dz}$$

- Shear strains

$$\gamma_{xy} = \gamma_{yx} = \frac{dU}{dy} + \frac{dV}{dx}$$ or $$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{dU}{dy} + \frac{dV}{dx}\right)$$

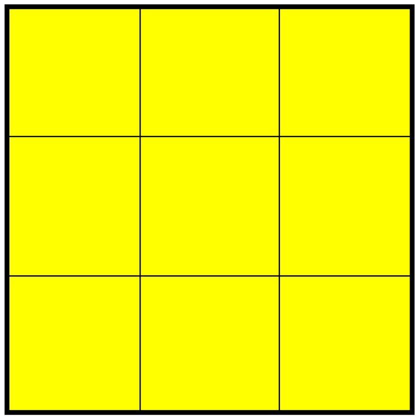
$$\gamma_{xz} = \gamma_{zx} = \frac{dU}{dz} + \frac{dW}{dx}$$ or $$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{dU}{dz} + \frac{dW}{dx}\right)$$

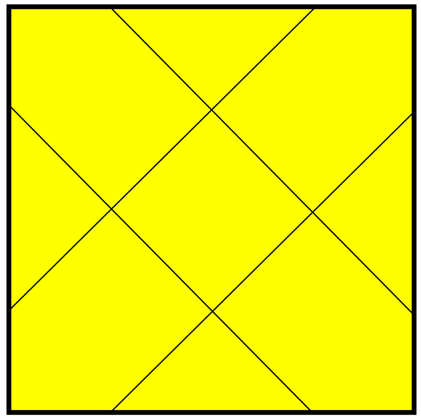
$$\gamma_{yz} = \gamma_{zy} = \frac{dV}{dz} + \frac{dW}{dy}$$ or $$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{dV}{dz} + \frac{dW}{dy}\right)$$

ひずみテンソル
Strain tensor

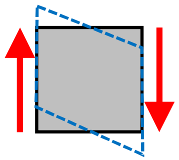
$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

座標変換・Coordinate Transformation

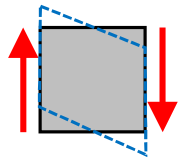




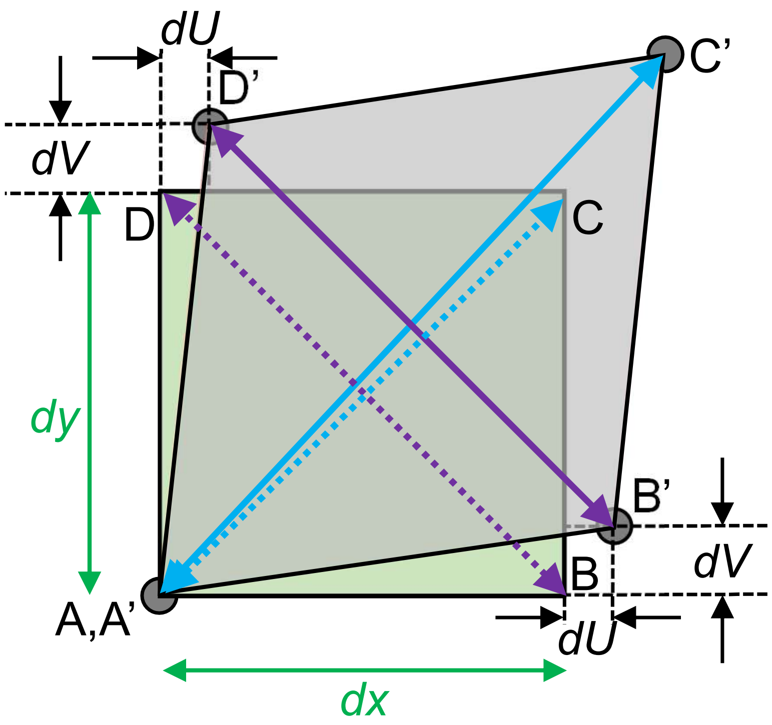
Shear
せん断



Shear
せん断



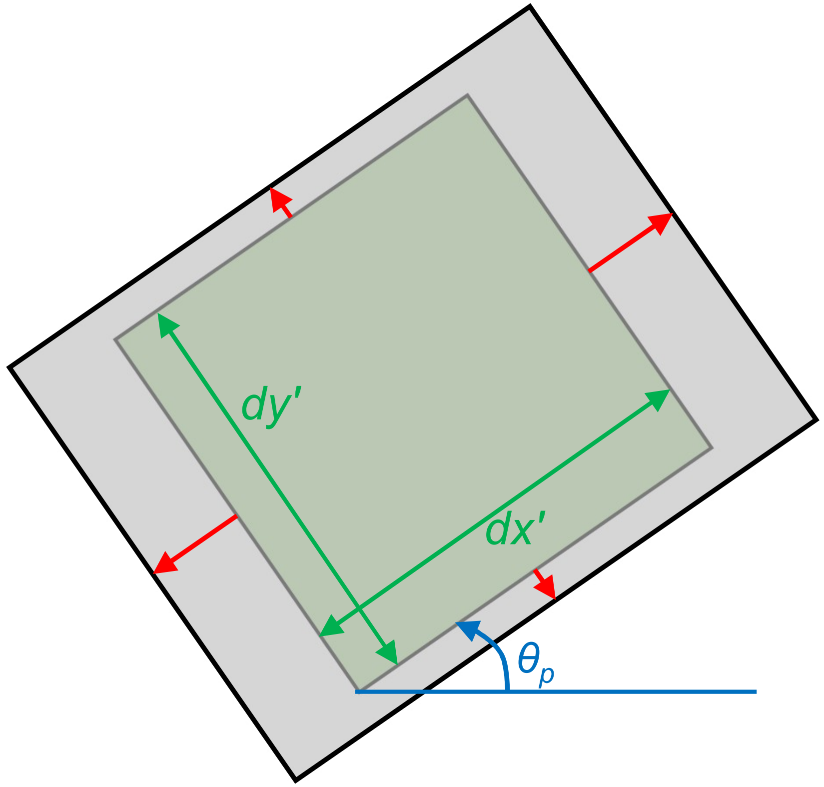
座標変換・Coordinate Transformation



L'_{AC}-L_{AC} > L'_{BD}-L_{BD}

座標を変換すると、垂直ひずみや
せん断ひずみが大きくなる可能性がある

If using a different coordinate system, the normal and shear strains will change



せん断ひずみがない座方がある

There exists an orientation where there is no shear strain

ひずみの主軸・Principal axis of strain

座標変換・Coordinate Transformation

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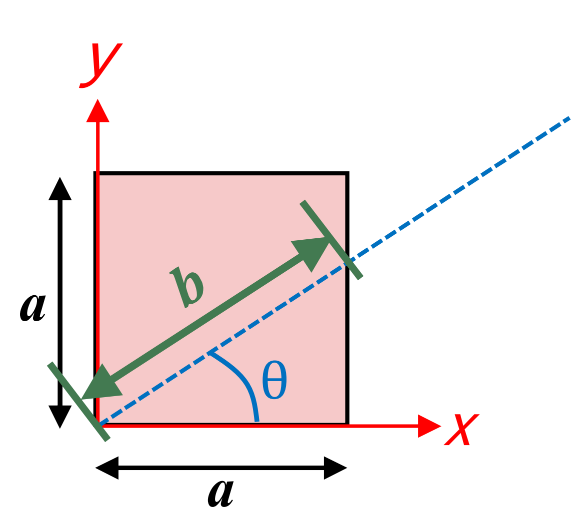
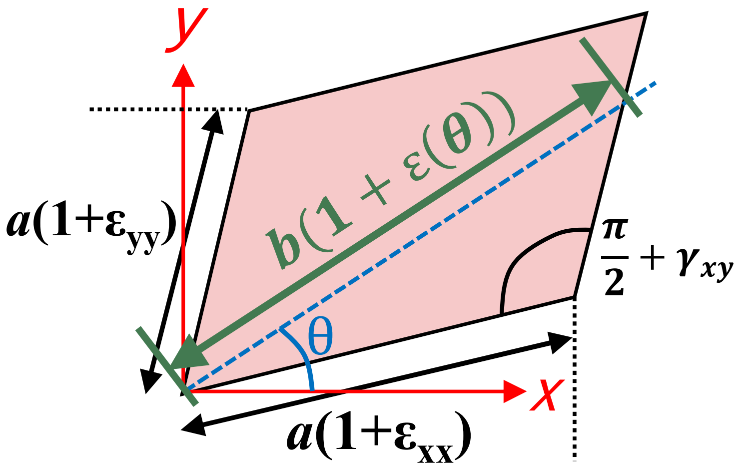
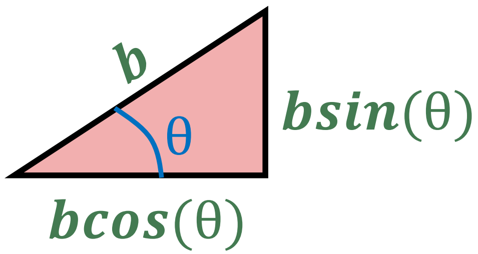
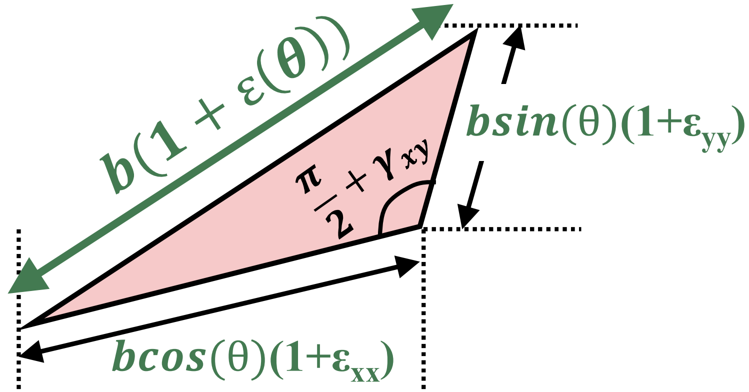
	元のブロック **Original shape**	変形したブロック **Deformed shape**
Full block		
Cut at angle *θ*		

座標変換・Coordinate Transformation

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- 余弦定理・Law of cosines:

$$(A'B')^2 = (A'C')^2 + (B'C')^2 - 2(A'C')(B'C')\cos(\Delta A'C'B')$$

- Assuming that ε and γ are extremely small:

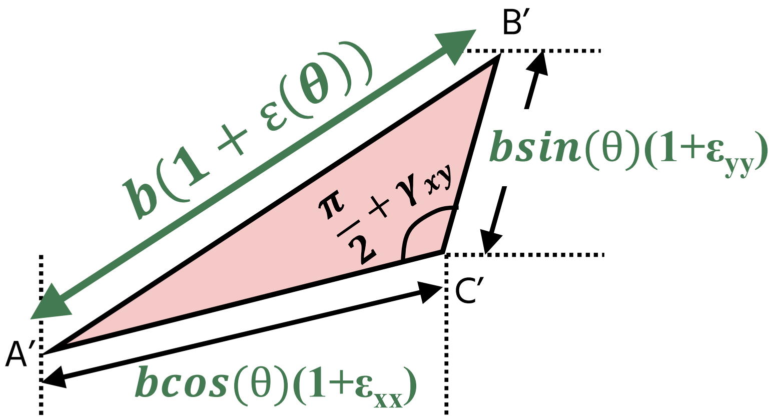
$$\cos\left(\frac{\pi}{2} + \gamma_{xy}\right) = -\sin(\gamma_{xy}) \approx -\gamma_{xy}$$

and

$$\varepsilon^2(\theta) \approx \varepsilon_{xx}^2 \approx \varepsilon_{yy}^2 \approx \gamma_{xy}^2 \approx \varepsilon_{xx}\varepsilon_{yy} \approx \varepsilon_{xx}\gamma_{xy} \approx \varepsilon_{yy}\gamma_{xy} \approx \varepsilon_{xx}\varepsilon_{yy}\gamma_{xy} \approx 0$$

- Substituting into law of cosine equation:

$$(b(1+\varepsilon(\theta)))^2 \approx (b\cos(\theta)(1+\varepsilon_{xx}))^2 + \left(b\sin(\theta)(1+\varepsilon_{yy})\right)^2 - 2(b\cos(\theta)(1+\varepsilon_{xx}))\left(b\sin(\theta)(1+\varepsilon_{yy})\right)(-\gamma_{xy})$$



座標変換・Coordinate Transformation

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- Expanding equation and dividing both sides by b^2:

$$1 + 2\varepsilon(\theta) + \varepsilon^2(\theta) = cos^2(\theta)(1+2\varepsilon_{xx}+\varepsilon_{xx}^2) + sin^2(\theta)(1+2\varepsilon_{yy}+\varepsilon_{yy}^2) + 2cos(\theta)sin(\theta)(\gamma_{xy}+\varepsilon_{xx}\gamma_{xy}+\varepsilon_{yy}\gamma_{xy}+\varepsilon_{xx}\varepsilon_{yy}\gamma_{xy})$$

- Setting blue terms to be approximately zero and expanding further

$$1 + 2\varepsilon(\theta) = cos^2(\theta) + sin^2(\theta) + 2cos^2(\theta)\varepsilon_{xx} + 2sin^2(\theta)\varepsilon_{yy} + 2cos(\theta)sin(\theta)\gamma_{xy}$$

- Recalling that 1 = cos²(θ)+sin²(θ)

$$\boldsymbol{\varepsilon(\theta) = cos^2(\theta)\varepsilon_{xx} + sin^2(\theta)\varepsilon_{yy} + cos(\theta)sin(\theta)\gamma_{xy}}$$

- Considering strain θ= 45° (sin(45°)=cos(45°)=$1/\sqrt{2}$):

$$\varepsilon_{45} = cos^2(45^o)\varepsilon_{xx} + sin^2(45^o)\varepsilon_{yy} + cos(45^o)sin(45^o)\gamma_{xy} = \frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy} + \gamma_{xy})$$

$$\gamma_{xy} = 2\varepsilon_{45} - (\varepsilon_{xx} + \varepsilon_{yy})$$

この関係は、ｘ軸方向に対するひずみを考慮する場合にも当てはまる

This relationship is true when considering strains relative to the x-axis

座標変換・Coordinate Transformation

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- Trigonometric functions:

$$cos^2(\theta) = \frac{1+cos(2\theta)}{2} \qquad sin^2(\theta) = \frac{1-cos(2\theta)}{2} \qquad cos(\theta)sin(\theta) = \frac{sin(2\theta)}{2}$$

- Substituting into strain equation:

$$\varepsilon(\theta) = cos^2(\theta)\varepsilon_{xx} + sin^2(\theta)\varepsilon_{yy} + cos(\theta)sin(\theta)\gamma_{xy}$$

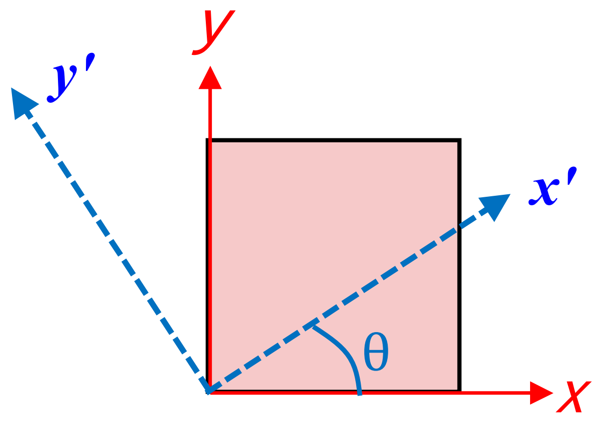
$$\varepsilon(\theta) = \left(\frac{1+cos(2\theta)}{2}\right)\varepsilon_{xx} + \left(\frac{1-cos(2\theta)}{2}\right)\varepsilon_{yy} + \frac{sin(2\theta)}{2}\gamma_{xy}$$

- Rearranging and setting $\varepsilon(\theta) = \varepsilon'_{xx}$:

$$\boldsymbol{\varepsilon'_{xx} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} cos(2\theta) + \frac{\gamma_{xy}}{2} sin(2\theta)}$$

- Rearranging so that terms without so that terms without sin(2θ) and cos(2θ) are on left:

$$\boldsymbol{\varepsilon'_{xx} - \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} = \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} cos(2\theta) + \frac{\gamma_{xy}}{2} sin(2\theta)}$$



座標変換・Coordinate Transformation

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- Consider an angle at 45° anticlockwise of x':

$$\varepsilon(\theta + 45^o) = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} cos\big(2(\theta + 45^o)\big) + \frac{\gamma_{xy}}{2} sin(2(\theta + 45^o))$$

$$= \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} sin(2\theta) + \frac{\gamma_{xy}}{2} cos(2\theta)$$

- Recall the relationship that $\gamma_{xy} = 2\varepsilon_{45} - \left(\varepsilon_{xx} + \varepsilon_{yy}\right)$ [Slide 13]. This must be true for any axis x'.

$$\gamma'_{xy} = 2\varepsilon(\theta + 45^o) - \left(\varepsilon'_x + \varepsilon'_y\right)$$

$$= 2\left(\frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} - \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2} sin(2\theta) + \frac{\gamma_{xy}}{2} cos(2\theta)\right) - \left(\varepsilon'_x + \varepsilon'_y\right)$$

- Simplifying, we get:

$$\boldsymbol{\frac{\gamma'_{xy}}{2} = \frac{-\left(\varepsilon_{xx} - \varepsilon_{yy}\right)}{2} sin(2\theta) + \frac{\gamma_{xy}}{2} cos(2\theta)}$$

座標変換・Coordinate Transformation

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- From slides 13 and 14:

$$\varepsilon'_{xx} - \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} = \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}cos(2\theta) + \frac{\gamma_{xy}}{2}sin(2\theta)$$

$$\frac{\gamma'_{xy}}{2} = \frac{-(\varepsilon_{xx}-\varepsilon_{yy})}{2}sin(2\theta) + \frac{\gamma_{xy}}{2}cos(2\theta)$$

Note that the ½ term is retained so that right side of equation has similar forms

- Squaring equations and summing together:

$$\left(\varepsilon'_{xx} - \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma'_{xy}}{2}\right)^2 = \left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}cos(2\theta) + \frac{\gamma_{xy}}{2}sin(2\theta)\right)^2 + \left(-\frac{(\varepsilon_{xx}-\varepsilon_{yy})}{2}sin(2\theta) + \frac{\gamma_{xy}}{2}cos(2\theta)\right)^2$$

- Expanding and simplifying becomes:

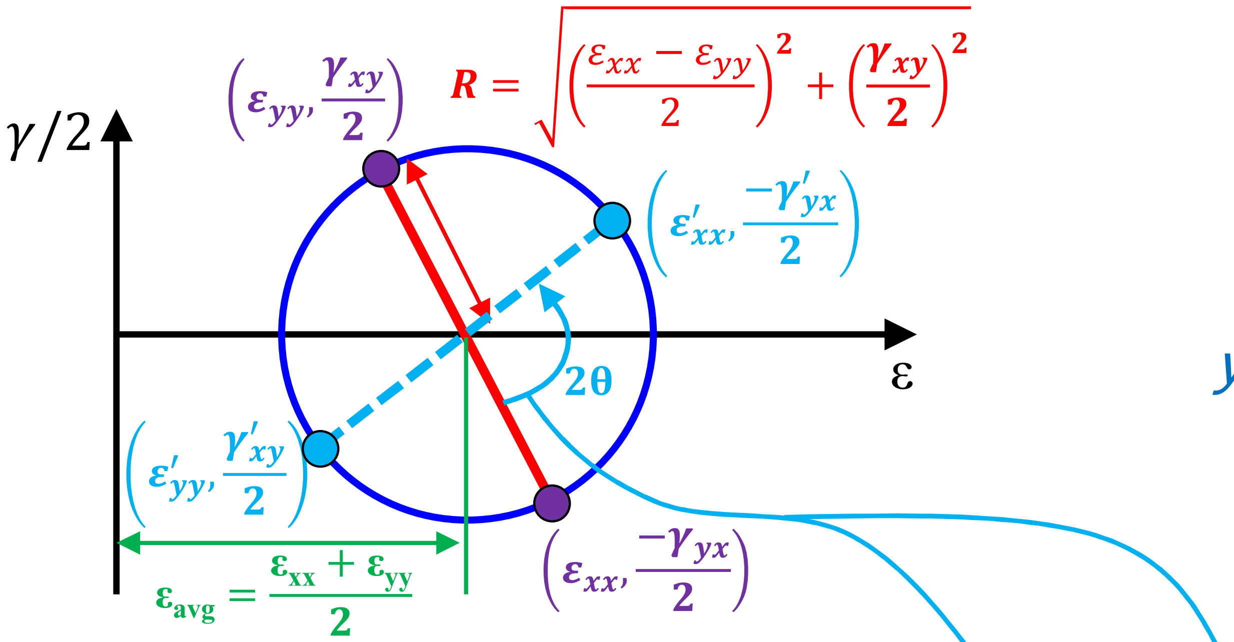
$$\left(\varepsilon'_{xx} - \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma'_{xy}}{2}\right)^2 = \left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2$$

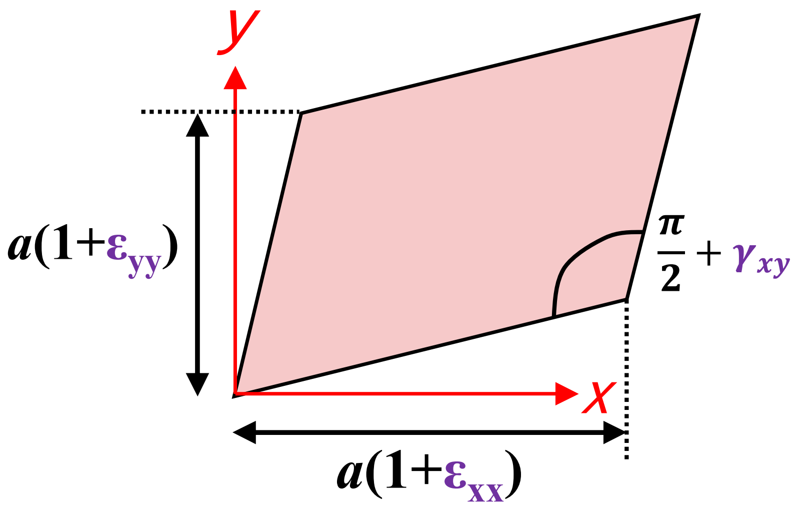
モールのひずみ円・Mohr's Strain Circle

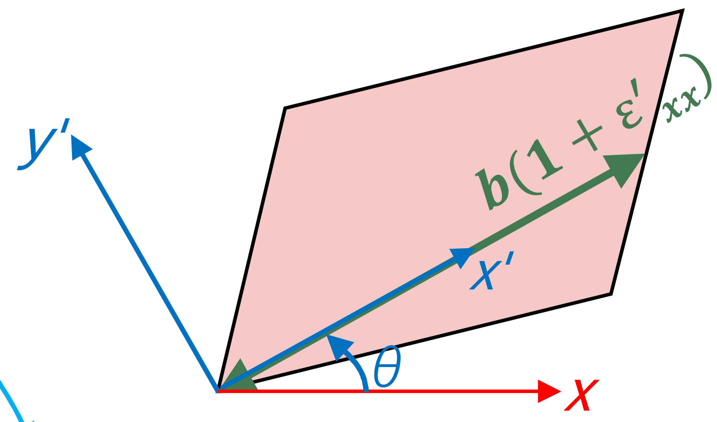
- This is in the form of an equation of a circle:

$$\left(\varepsilon'_{xx} - \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma'_{xy}}{2}\right)^2 = \left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2$$

$$(X - A)^2 + (Y - B)^2 = R^2 \quad (B \to 0)$$







If $\varepsilon_{xx} > \varepsilon_{yy}$

$$\varepsilon'_{xx} - \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} = \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2} cos(2\theta) + \frac{\gamma_{xy}}{2} sin(2\theta)$$

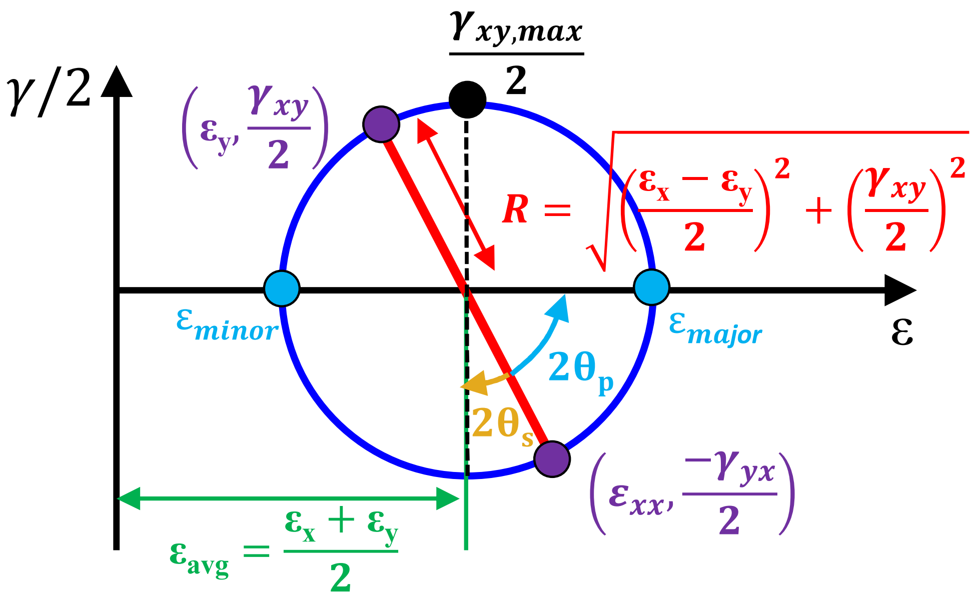
主ひずみ・Principal Strain

- Principal strains (i.e., where $\gamma'_{xy} = \gamma'_{yx} = 0$):

$$\varepsilon_{major} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \varepsilon_{mino} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} - \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \tan(2\theta_p) = \frac{\gamma_{xy}}{\varepsilon_{xx}-\varepsilon_{yy}}$$

- Maximum shear strain:

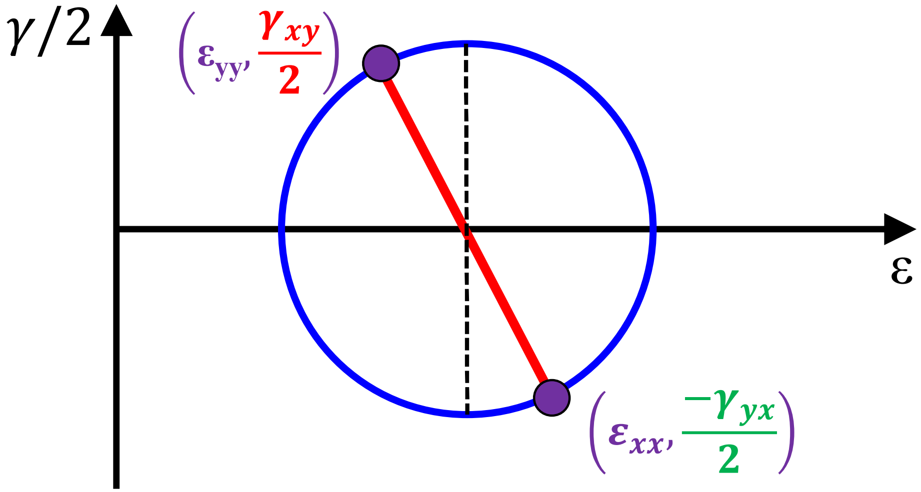
$$\gamma_{xy,max} = 2\sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \tan(2\theta_s) = \frac{\varepsilon_{xx}-\varepsilon_{yy}}{\gamma_{xy}}$$

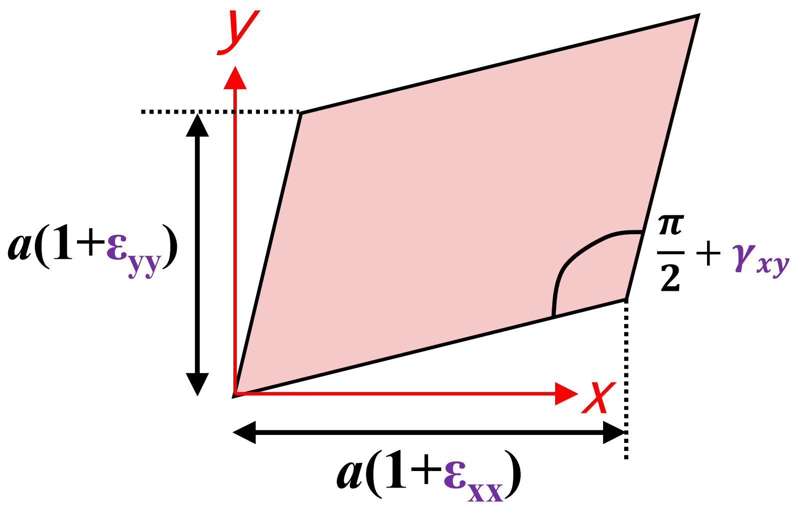


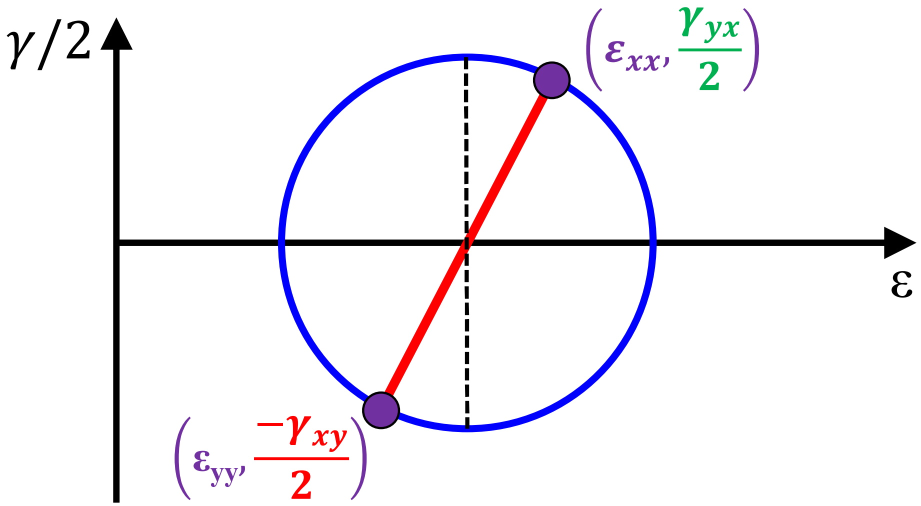
We will revisit transformation of strains in future classes

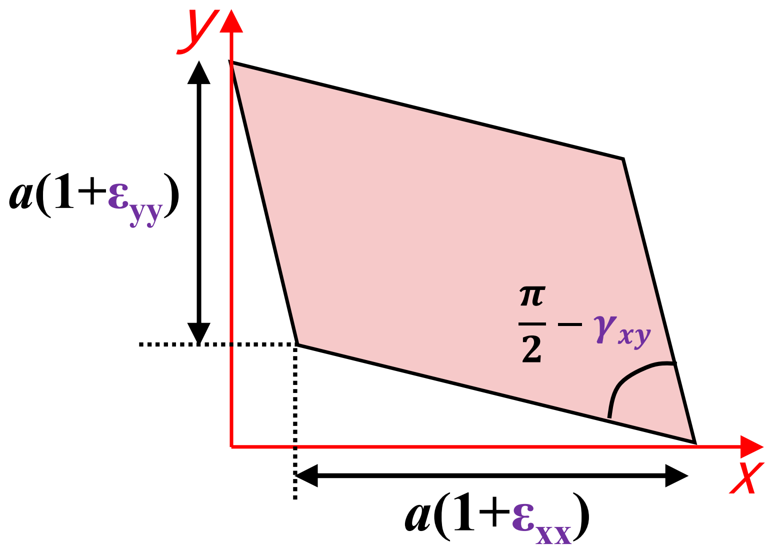
ひずみの変換について、今後の授業で再度取り上げる予定である

符号の規約・Sign Convention









例）主ひずみ・Principal Strain

For the following strains, determine the principal strains and angle to principal axis:

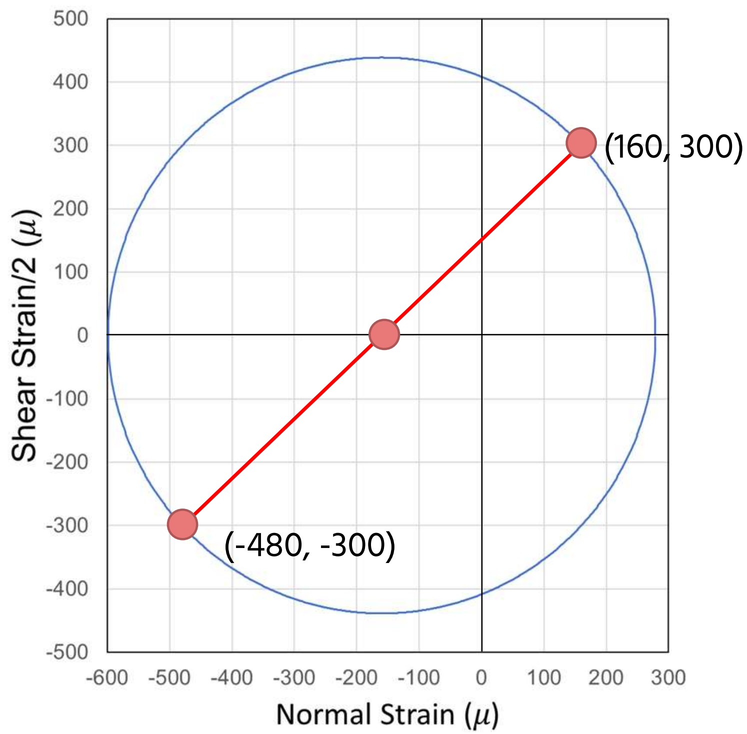
ε_{xx}=+160μ (+160×10^{-6}), ε_{yy}=-480μ, γ_{xy}=-600μ

Principal strains:

$$\varepsilon_{major,minor} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Angle to principal axis:

$$\tan(2\theta_p) = \frac{\gamma_{xy}}{\varepsilon_{xx}-\varepsilon_{yy}}$$



例）最大せん断ひずみ・Maximum shear strain

For the following strains, determine the principal strains and angle to principal axis:

ε_{xx}=+160μ (+160×10^{-6}), ε_{yy}=-480μ, γ_{xy}=-600μ

Maximum shear strain:

$$\gamma_{xy,max} = 2\sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$